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Result Involution Graphs of Finite Groups

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Abstract

In this paper, a new kind of graph on a finite group G, namely the result involution graph is defined and studied. We use Γ_G^{RI} to denote this graph, is a simple undirected graph with vertex set G. Two distinct vertices are adjacent if and only if their product is nontrivial involution element in G. The result involution graph for several finite groups are obtained. We study some properties of the result involution graph Γ_G^{RI} by resizing graph by using the conjugacy classes of G. Finally, we show that the result involution graphs for the symmetric groups S_n and the alternating groups A_n are connected with diameter at most 3 and radius at most 2 for $n > 4$. Furthermore, they have girth 3.

1. Introduction

An effective method to analyze the structure of a group is studying the group action on a graph that is leading to many fascinating results. Since this method is most active research area nowadays, there are enormous research papers in this area. Studies on commuting and non-commuting involution graphs have been conducted by many authors in [1, 4, 7].

In addition, in [2] authors defined the commuting involution graph for symmetric groups whose vertex set contains all involution elements and two vertices are adjacent if they commute. Moreover, involution graphs, which adjacency of two vertices is defined by their product is of order 3, are defined in [3]. In the recent research, Tolue [8] introduced the twin non-commuting graph of a group by partitioning the vertex set of non-commuting graph with special properties. Assume that G is a finite group that acts on a finite set X. The orbit of $x \in X$ is the set $O(x) = \{g * x : g \in G\}$. In this paper, we consider the conjugation action so the orbit can be written as $x^G = \{x^g : g \in G\}$. In particular if G acts on itself by conjugation, then it is called a conjugacy class of x. The conjugacy classes partition G into some equivalence classes. In other words, every element in a group lies exactly in one conjugacy class. The conjugacy classes of a finite group are well known in representation theory. According to Atlas notation, we are going to use the symbol d_iA where d_i is order of x_i of type A and so on [5]. A non trivial element x, $(x \neq e)$, in a group G is called involution element if $x^2 = e$. The set of all involution elements in G is denoted by $I(G)$. We deduce that the size of $I(G)$ is the sum of the size of conjugacy classes of involutions and denoted by t. The presentations of the dihedral groups D_{2n} and the generalized quaternion groups Q_{4n} are

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1, y^{-1}xy = x^{-1} \rangle,$$

$$Q_{4n} = \langle a, b \mid a^n = b^2, \quad a^{2n} = 1, b^{-1}ab = a^{-1} \rangle,$$

and

$$C_n = \langle a : a^n = e \rangle.$$

In this paper, the result involution graph based on involution elements in a group and only we consider finite groups. We find the structure of graphs associated to these groups. Some properties of these graphs have been studied such as connectedness, diameter, and girth.

The following results are well known in group theory.

Lemma 1.1. Let $G = D_{2n}$ be the Dihedral group. Then the number of involution elements in G is

$$t = \begin{cases} |2A| = n & if\ n\ is\ odd \\ |2A| + |2B| + |2C| = \frac{n}{2} + \frac{n}{2} + 1 & if\ n\ is\ even \end{cases}$$

Lemma 1.2. There is a unique conjugacy class of involution in Q_{4n}.

Lemma 1.3. The number of elements of order four in D_{2n} is two if $n \equiv 0\ mod\ 4$, otherwise is zero.

Proof. We know that $G = D_{2n}$ generated by two elements a and b where $a^n = e = b^2$ and $bab^{-1} = a^{-1}$. Since $n = 4k$, then by [5], there are $2k + 3$ conjugacy classes ofD_{2n} as follows: $e^G = \{e\}$, $(a^{2k})^G = \{a^{2k}\}, (a^i)^G = \{a^i, a^{-i}\}$ for $i = 1,2, \dots ,2k - 1, (a^{2j+1}b)^G = \{a^{2j+1}b: j = 0,1, \dots ,2k - 1\}$ and $(a^{2j}b)^G = \{a^{2j}b: j = 0,1, \dots ,2k - 1\}$. They have size 1,1, $2(2k - 1), 2k$ and $2k$ respectively. In particular, $(a^k)^G = \{a^k, a^{-k}\}$ is the conjugacy class of order four of size two. If $n \neq 4k$, then 4 does not divide n. Therefore it has not elements of order 4.

Lemma 1.4. For any $G = Q_{4n}$, the number of elements of order four is

$$F = \begin{cases} 2n & if\ n\ is\ odd \\ 2n + 2 & if\ n\ is\ even \end{cases}$$

Proof. Every element of Q_{4n} can be written as $a^p b^q$ where $0 \leq p < 2n$ and $q \in \{0,1\}$. If a^p is an element of order four then $p = \frac{n}{2}, \frac{3n}{2}$, that means, for n is odd there is no such elements. If n is even, then it gives two elements of order four. Now we consider $(a^p b)^4$. One can apply the relation $ab = ba^{-1}$ to get the identity element that is, there is $2n$ elements of order four for any n. This completes the proof.

2. Some Properties of Result Involution Graphs

Now we are going to introduce a new type of graph which is called the result involution graph as follows:

Definition 2.1. Let G be a finite group of even order, we assign a graph which is called a result involution graph and denoted by Γ_G^{RI}, whose vertex set is the whole group G and two distinct vertices are adjacent if their product is an involution in G.

We now do for a couple of examples, however this not practical method for studying the result involution graph in general.

Example 2.2. The result involution graph of D_8 is

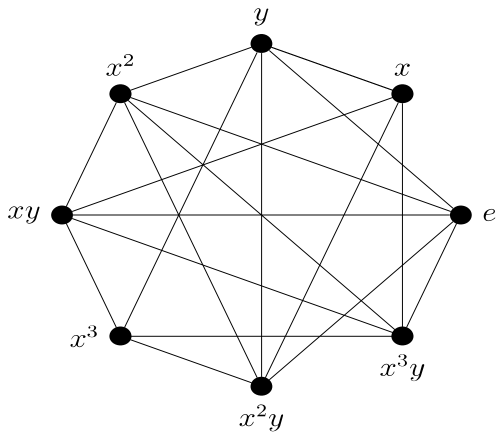

Figure 1: The Result involution graph of D_8

Example 2.3. Consider the alternating group $G = A_4$. It well known that $A_4 = \langle a, b | a^3 = b^3 = e, aba = b^{-1}\rangle$. It has three involution elements lie in one class of type $2A = \{ab, a^2b^2, ab^2a\}$. Also, it has two classes of elements of order three such as $3A = \{a, a^2, a^2b, ba^2\}$ and $3B = \{b, b^2, b^2a, ab^2\}$. It is clear that the product of the identity element with elements of $2A$ give involution elements. This leads the complete graph K_4. Furthermore, the product among elements of $3A$ and $3B$ give some involution elements such elements are $[a^2, ba^2], [a^2, a^2b], [b^2, b^2a]$ and $[ab^2, b^2]$. On the other hand, the product of elements of $3A$ with $3B$ which give an involution element in A_4 are $[a, b], [a, b^2a], [a, ab^2], [a^2, b^2], [a^2b, b], [b, ba^2], [a^2b, ab^2], [a^2b, b]$ and $[b^2a, ba^2]$. From these, the result involution graph, which is $\Gamma_{A_4}^{RI} = K_4 \cup \{3 - \text{regular on 8 vertices}\}$ can be drawn in Figure 2. It has girth 3.

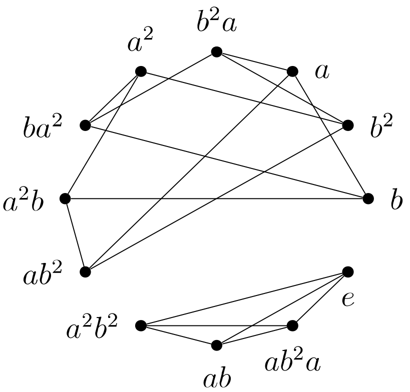

Figure 2: Result Involution Graph of A_4.

Proposition 2.4. Let G be a finite group with t involution, then the number of edges in Γ_G^{RI} is $\frac{1}{2}(t|G| - F)$ where F is the number of elements of order 4.

Proof. Given any two distinct elements g, h in G are joined by an edge if $gh = x$ for some involution element x. That is $h = g^{-1}x$. If g does not have order 4. Then the degree of it is t. If g has order 4 then the degree of it is $t - 1$ because g^2 is an involution by taking $x = g^2$, this force $h = g$ which is impossible by assumption. From this we conclude that the number of edges in the graph Γ_G^{RI} is $\frac{1}{2}(t|G| - F)$, where F is the number of elements of order 4.

The following results can be obtained from Proposition 2.4.

Corollary 2.5. Let Γ_G^{RI} be a result involution graph of the group $G = D_{2n}$. Then the number of edges is

$$|E(\Gamma_G^{RI})| = \begin{cases} n^2 & if\ n\ is\ odd \\ n^2+n & if\ n\ is\ even\ and\ 4 \nmid n \\ n^2+n-1 & if\ n\ is\ even\ and\ 4 \mid n. \end{cases}$$

Proof. If n is odd, then by Lemma 1.1, $t = n$ and Lemma1.3, $F = 0$. Thus
$| E(\Gamma_G^{RI})| = \frac{1}{2}(n(2n) - 0) = n^2$.
If n is even and $4|n$, then by Lemma1.1, $t = n + 1$ and Lemma1.3, $F = 0$. Thus
$| E(\Gamma_G^{RI})| = \frac{1}{2}((n+1)(2n) - 0) = n^2 + n$.
If n is even and $4|n$, then by Lemma1.1, $t = n + 1$ and Lemma1.3, $F = 2$. Thus
$|E(\Gamma_G^{RI})| = \frac{1}{2}((n+1)(2n) - 2) = n^2 + n - 1$.

Remark 2.6. For any vertex $v \in \Gamma_{D_{2n}}^{RI}$ and n is even, $deg(v) = \begin{cases} n, & if\ O(v) = 4 \\ n+1, & if\ O(v) \neq 4 \end{cases}$ and $deg(v) = n$, if n is odd.

Lemma 2.7. The result involution graph of D_{2n} is the complete bipartite graph, $K_{n,n}$, if n is odd.

Proof. Since there are n involution elements and they are connected to remaining elements then the degree of all vertices is n. In addition, in general product of two involution elements is not an involution in D_{2n} then they are not connected to each other and non-involution elements also are not connected to each other. Thus, there are two parts and the degree of each vertex is n, that is $K_{n,n}$ which is also a regular graph.

Proposition 2.8. The result involution graph of D_{2n} is connected.

Proof. If n is odd, then by the above lemma, its result involution graph is the complete bipartite graph, $K_{n,n}$, then it is connected. If n is even, then by Lemma 1.1, there are $n + 1$ involution elements. Since each x^i is connected to all yx^j for $j \in \{1,2,\dots,n-1\}$, yx^j and $yx^{\frac{n}{2}+j}$ are connected for $j \in \{0,1,2,\dots,n-1\}$ then the graph is connected. Hence, the result follows.

Lemma 2.9. The girth of result involution graph of D_{2n} is 3 if n is even and 4 if n is odd.

Proof. The girth is the length of the smallest cycle in a simple graph. If n is odd, then the graph is complete bipartite and the length of any cycle is four. If n is even, then one can construct a cycle of length three by $yx^i, x^{\frac{n}{2}}$ and the identity element of D_{2n}.

Lemma 2.10. The chromatic number of the result involution graph of D_{2n} is 2 if n is odd and 4 if n is even.

Proof. If n is odd then the result involution graph is a complete bipartite graph. Thus, it can be colored by two colors. If n is even then we can consider $yx^{\frac{n}{4}}, x^{\frac{n}{2}}, yx^{\frac{3n}{4}}$ and the identity element of D_{2n} and since each pair of these elements is connected, hence the chromatic number is 4.

Lemma 2.11. The diameter of the result involution graph of D_{2n} is two.

Proof. If n is odd, then by Lemma 2.7 the result involution graph of D_{2n} is the complete bipartite graph, $K_{n,n}$. Hence its diameter is two.

For n is even, consider x^i and x^j for any $i, j \notin \{\frac{n}{4}, \frac{3n}{4}\}$. Since these two elements are not connected to each other and they are connected to other involution elements of the form yx^i for any i, then the maximum distance is 2.

Proposition 2.12. Let Γ_G^{RI} be a result involution graph of the group $G = Q_{4n}$, then the number of edges is

$$|E(\Gamma_G^{RI})| = \begin{cases} n & if\ n\ is\ odd \\ n-1 & if\ n\ is\ even \end{cases}$$

Proof. It follows from Proposition 2.4 and Lemma 1.4.

Proposition 2.13. The graph $\Gamma_{Q_{4n}}^{RI}$ is disconnected.

Proof. By Proposition 2.12, we have n edges if n is odd. By hand shaking lemma $\sum_{x\in Q_{4n}} deg(x) = 2|E(\Gamma_{Q_{4n}}^{RI})| = 2n$. If we partition $2n$ over the number of vertices, which is $4n$, we have some vertices with no edges that is degree of those vertices is zero. Hence, it is disconnected.

Remark 2.14. If n is odd, then the result involution graph for cyclic group C_n is a null graph, that is $N_{|G|}$, as the given group has no involution element. If n is even and $n > 2$, then the result involution graph for a cyclic group C_n is disconnected, because it has only one involution element. Finally, the result involution graph for the cyclic group C_2 is K_2.

Finally, we present some results about symmetric and alternating groups of degree n. To do this, since the order of these groups are large if n is large enough. For instance, the symmetric group S_{10} has order 3628800, the result involution graph has 3628800 vertices and 17227623420 edges (use Proposition 2.4). Now we use a new idea to decide whether the result involution graph is connected or not. The idea is to resize the result involution graph by reducing the size of the vertex set. Thus, we use the conjugacy classes of a group instead of all elements of it. Therefore, we only use one representative element form each of the conjugacy class of a group G. Two conjugacy classes are connected if there exists an edge between the representative elements of them. By using this idea, the resize graph has only 42 vertices and 699 edges Let us start with the following result.

Lemma 2.15. The resized graph of Γ_G^{RI} for a finite group G is disconnected if and only if the result involution graph Γ_G^{RI} is also disconnected.

Proof. Since the resized graph of Γ_G^{RI} is disconnected, then there exists at least one conjugacy classes of G which has no edges with other conjugacy classes. So, the elements of this conjugacy class have no edge with other elements. Thus, a result involution graph Γ_G^{RI}, is also disconnected. Conversely, since the result involution graph is disconnected (conjugacy classes), then there are at least two connected components. For any connected component, one can choose a vertex as representative for its conjugacy class. These representatives are not adjacent. Thus, the result follows.

Example 2.16. The resized graph of the result involution graph of A_4 is two copies of K_2 which is present in Figure 3.

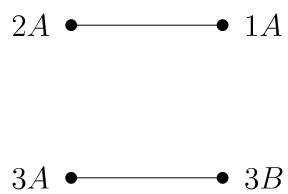

Figure 3: Resized Graph of $\Gamma_{A_4}^{RI}$.

Example 2.17. From Table 1, the symmetric group S_4 has 9 involution elements which are in 2A and 2B and 105 edges which is equal to the sum of upper triangle without size of conjugacy classes. It is hard to draw its graph. So, we draw the resized graph of it just by taking one representative from each pair of conjugacy class and joining them together according Table 1 if it exists.

Table 1: Vertex set and Edge set of $\Gamma_{S_4}^{RI}$.

Conjugacy Classes	1A	2A	2B	3A	4A
1A	0	6	3	0	0
2A		3	6	24	12
2B			3	0	12
3A				12	24
4A					0

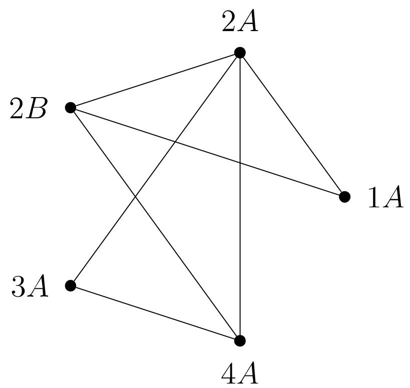

Figure 4: Resized Graph of $\Gamma_{S_4}^{RI}$

Proposition 2.18. If G is the alternating group of degree n or G is the symmetric group of degree n, where $n > 4$ then the result involution graph of G is connected with diameter at most 3 and radius at most 2.

Proof. The proof follows from Lemma 2.15.

Proposition 2.19. If G is the alternating group of degree n or G is the symmetric group of degree n, where $n > 4$ then the result involution graph of G has girth 3.

Proof. Since $n > 3$, then it is clear that the trivial class adjacent with involution classes. There are elements in different involution classes which are adjacent together. This produces K_3 as a subgraph of the result involution graph. Hence it has girth 3.

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